

## 4.4 - Linear Independence

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**Definition:** If  $S = \{v_1, v_2, \dots, v_r\}$  is a set of two or more vectors in a vector space  $V$ , then  $S$  is said to be a **linearly independent set** if no vector in  $S$  can be expressed as a linear combination of the others. A set that is not linearly independent is said to be **linearly dependent**. If  $S$  has only one vector, we will agree that it is linearly independent if and only if that vector is nonzero.

$\{\hat{i}, \hat{j}\} = \{\vec{e}_1, \vec{e}_2\}$  is a lin. indep. set in  $\mathbb{R}^2$ .

$\{\hat{i}, \hat{j}, (3, 2)\}$  is not, since  $3\hat{i} + 2\hat{j} = (3, 2)$

**Theorem 4.4.1** A nonempty set  $S = \{v_1, v_2, \dots, v_r\}$  in a vector space  $V$  is linearly independent if and only if the only coefficients satisfying the vector equation  $k_1 v_1 + k_2 v_2 + \dots + k_r v_r = \vec{0}$  are  $k_1 = 0, k_2 = 0, \dots, k_r = 0$ .

$$0\hat{i} + 0\hat{j} = \vec{0} \quad -3\hat{i} - 2\hat{j} + (3, 2) = \vec{0}$$

pf: ( $\Rightarrow$ ) Let  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$  be a lin.

indep. set. Suppose  $\exists k_1, k_2, \dots, k_r$  not all zero

$\exists k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r = \vec{0}$ . If  $k_i \neq 0$ ,

then  $k_i \vec{v}_i = -k_1 \vec{v}_1 - \dots - k_r \vec{v}_r \Rightarrow \vec{v}_i = -\frac{k_1}{k_i} \vec{v}_1 - \dots - \frac{k_r}{k_i} \vec{v}_r$

a linear combination of the remaining vectors.

This contradicts linear independence.

( $\Leftarrow$ ) Suppose  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$  is not a lin.

independent set. Then  $\vec{v}_i = c_1 \vec{v}_1 + \dots + c_r \vec{v}_r$

for some  $i$ .  $\Rightarrow \vec{v}_i - c_1 \vec{v}_1 - \dots - c_r \vec{v}_r = \vec{0}$ ,  
contradicting that all the coefficients  
are zero. ✓

Proof by contradiction: if  $\Rightarrow$  then. "If" is true.

Assume the opposite of "then" & show this  
contradicts "If".

#3 In each part, determine whether the vectors are linearly independent or are linearly dependent in  $\mathbb{R}^4$ .

a.  $(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (4, 2, 6, 4)$

b.  $(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)$

a) The set is indep. if the only way  
 $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_4 \vec{v}_4 = \vec{0}$  is if  $c_i = 0$   
for all  $i$ .

that is, does  $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \ | \ \vec{0}]$  have  
a solution other than the trivial solution.

Recall that  $A\vec{x} = \vec{0}$  has only the  
trivial solution iff

$A$  is invertible  $\Leftrightarrow \text{ref}(A) = I_n$   
 $\Leftrightarrow \det(A) \neq 0$ .

$$\begin{array}{c} \text{row} \\ \text{1} \\ \text{2} \\ \text{3} \end{array} \left| \begin{array}{cccc} 1 & 2 & 4 & 4 \\ 5 & -1 & 2 & 2 \\ 3 & 2 & 6 & 6 \\ -3 & -1 & 6 & 4 \end{array} \right| = \begin{array}{c} \text{if turns} \\ \text{out} \\ \text{=} \textcircled{0} \end{array} \left| \begin{array}{cccc} 0 & 0 & 8 & 8 \\ 8 & 5 & -1 & 2 \\ 5 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{array} \right|$$

So the system has nontrivial solutions.

So the set is not lin. indep.

b) You verify that  $\det(A) \neq 0$

So the set is indep.

#6 Determine all values of  $k$  for which the following matrices are linearly independent in  $M_{22}$ .

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

Can we find  $a, b, c$  (not all zero)  $\Rightarrow$

$$a \begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + b \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + c \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ?$$

$$a - b + 2c = 0$$

$$a + bk + c = 0$$

$$ak + b + 3c = 0$$

$$\begin{array}{l} \text{indep} \\ \text{iff} \end{array} \left| \begin{array}{ccc|c} 1 & -1 & 2 & \\ 1 & k & 1 & \neq 0 \\ k & 1 & 3 & \end{array} \right|$$

$$\left| \begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 1 & k & 1 & 1 & k \\ k & 1 & 3 & k & 1 \end{array} \right|$$

$$\Rightarrow 3k - k + 2 - 2k^2 - 1 + 3 = 0$$

$$\Rightarrow \dots (k-2)(k+1) = 0 \Rightarrow k = -1, 2$$

The set is lin. indep. if  $k \neq -1, 2$ .

#9

- a. Show that the three vectors  $\mathbf{v}_1 = (0, 3, 1, -1)$ ,  $\mathbf{v}_2 = (6, 0, 5, 1)$ ,  $\mathbf{v}_3 = (4, -7, 1, 3)$  form a linearly dependent set in  $\mathbb{R}^4$ .
- b. Express each vector in part (a) as a linear combination of the other two.

$$\begin{bmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ -1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -7/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) so  $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$  if  $c_1 = \frac{2}{3} c_3$   
and  $c_2 = -\frac{2}{3} c_3$ .  $c_3 = 3 \Rightarrow c_1 = 2, c_2 = -2$   
so  $2\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3 = \vec{0}$ . Solving for each  $\vec{v}_i$  satisfies (b).

#14 In each part, let  $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be multiplication by  $A$  and let  $\mathbf{u}_1 = (1, 0, 0)$ ,  $\mathbf{u}_2 = (2, -1, 1)$ , and  $\mathbf{u}_3 = (0, 1, 1)$ . Determine whether the set  $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$  is linearly independent in  $\mathbb{R}^3$ .

a.  $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

b.  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

Step 1:  $T_A(\vec{u}_1) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$T_A(\vec{u}_2) = A \vec{u}_2, \quad T_A(\vec{u}_3) = A \vec{u}_3$$

and then proceed as before.

**Theorem 4.4.2**

- a) A set with finitely many vectors that contains  $\mathbf{0}$  is linearly dependent.
- b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

**Theorem 4.4.3** Let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$  be a set of vectors in  $R^n$ . If  $r > n$ , then  $S$  is linearly dependent.

This essentially boils down to a matrix equation and a system that has more columns than rows and so contains free variables.

**Definition:** If  $f_1 = f_1(x)$ ,  $f_2 = f_2(x)$ , ...,  $f_n = f_n(x)$  are functions that are  $n-1$  times differentiable on the interval  $(-\infty, \infty)$ , then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of  $f_1, f_2, \dots, f_n$ .

**Theorem 4.4.4** If the functions  $f_1, f_2, \dots, f_n$  have  $n-1$  continuous derivatives on the interval  $(-\infty, \infty)$ , and if the Wronskian of these functions is not identically zero on  $(-\infty, \infty)$ , then these functions form a linearly independent set of vectors in  $C^{(n-1)}(-\infty, \infty)$ .

*If D, then q is not the same as if not P, then not q.*

**#19** Use the Wronskian to show that the following sets of vectors are linearly independent.

a.  $1, x, e^x$

b.  $1, x, x^2$

$$a \cdot 1 + b \cdot x + c \cdot e^x = 0$$

$$a) W(1, x, e^x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x \neq 0$$

so  $\{1, x, e^x\}$  is a lin. indep. set.

$$b) W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

Consider  $\{1, \cos 2x, \sin^2 x\}$ .

This is a dependent set.

$$\cos 2x = 1 - 2\sin^2 x \Rightarrow \cos 2x - 1 + 2\sin^2 x = 0$$

and

$$\begin{vmatrix} 1 & \cos 2x & \sin^2 x \\ 0 & -2\sin 2x & \sin 2x \\ 0 & -4\cos 2x & 2\cos 2x \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2\sin 2x & \sin 2x \\ -4\cos 2x & 2\cos 2x \end{vmatrix}$$

$$= -4\sin 2x \cos 2x + 4\cos 2x \sin 2x = 0$$